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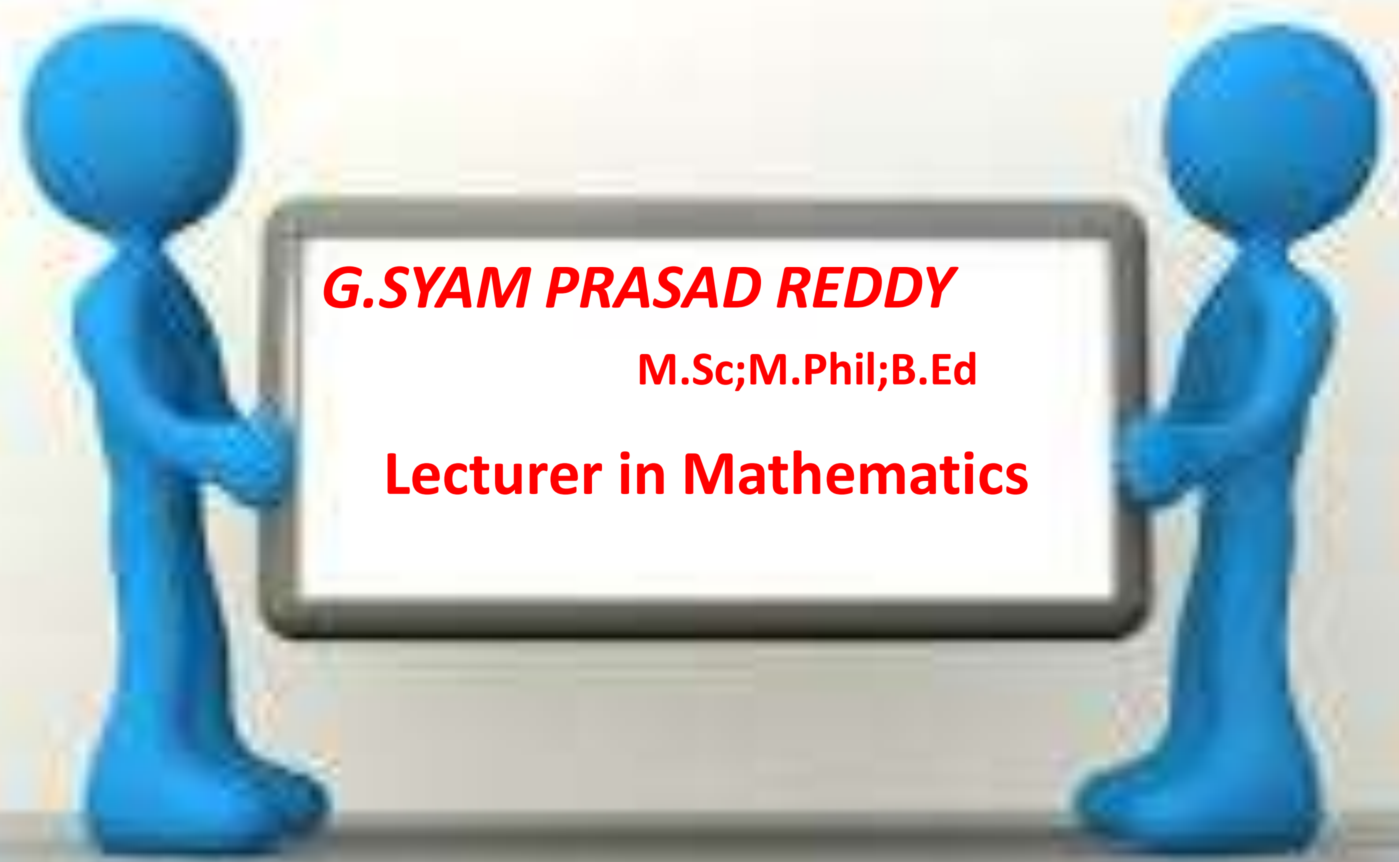
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SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Determination of roots of an equation of the form f(x)=0 has great importance in the fields of Science and Engineering. In this chapter we consider some simple methods of obtaining approximation roots of algebraic and transcendental equations.

- **Polynomial functions :**

  A function f(x) is said to be a polynomial function if f(x) is a polynomial in x. i.e $f(x)= a_0x^n+a_1x^{n-1}+.....+a_{n-1}x+a_n$, where $a_0 \neq 0$, the coefficients $a_{0,}a_{1,....}a_{n-1}\ a_n$ are real constants and n is a non-negative integer.

- **Algebraic function & Transcendental function :**

  A function which is a sum or difference or product of two polynomials is called an algebraic function. Otherwise the function is called a transcendental or non-algebraic function.

  If f(x) is an algebraic function then the equation f(x)=0 is called an algebraic equation.

  If f(x) is an transcendental function then the equation f(x)=0 is called an transcendental equation.

  e.g $f(x)=2\log x-(\prod/4)=0$, $f(x)= e^{5x}-x^3/2+3=0$ are examples of transcendental equations.

BISECTION METHOD

Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzano method. Some times it is referred to as half-interval method.

- Suppose we know an equation of $f(x)=0$ has exactly one real root between two real numbers $\mathbf{x_0}$, x_1. The numbers is choosen such that $f(\mathbf{x_0})$ and $f(x_1)$ will have opposite sign. Let us bisect the interval $(\mathbf{x_0}, x_1)$ into two half intervals and find the mid point

$\mathbf{x_2 = (x_0}+x_1)/2$.

If $f(x_2)=0$ then x_2 is a root .

If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between $\mathbf{x_0}$ **and** x_2.

The interval is taken as [$\mathbf{x_0}$, x_2] .Otherwise the root lies in the interval [$\mathbf{x_2}$, x_1]

Repeating the process of bisection we obtain successive subintervals which are smaller. At each iteration , we get the mid-point as a better approximation of the root . This process is terminated when interval is smaller than the desired accuracy. This is also called as " Interval Halving method".

1. Find a real root of the equation $f(x)= x^3-4x-9 =0$ using the Bisecting method in four stages.

Given $f(x)= x^3-4x-9 =0$

Now $f(2.7) = -0.117 = -ve$

and $f(2.8) = 1.752 = + ve$

Therefore the real root of the given equation lies in (2.7;2.8).

I) Let $a_0 = 2.7$ and $a_1 = 2.8$

By Bisecting method we have $\mathbf{X_n = (a_{n-1} + a_n) / 2.}$

So $X_1 = (a_0 + a_1) /2 = (2.7+2.8)/2 = 2.75$

Now $f(X_1)= 0.726874999 = + ve$

Therefore the root lies in (2.7;2.75)

II) Again let $a_1 = 2.7$ and $a_2 = 2.75$

So $X_2= (a_1 + a_2) / 2 = (2.7 + 2.75)/2 = 2.725$

Now $f (X_2) = 0.334828124 = + ve$

Therefore the root lies in (2.7 ; 2.725)

III) Let $a_2 = 2.7$ and $a_3 = 2.725$

So $X_3= (a_2 + a_3) / 2 = (2.7 + 2.725)/2 = 2.7125$

Now f (X_3) = 0.107642578 = + ve

Therefore the root lies in (2.7 ; 2.7125)

IV) Again let a_3 = 2.7 and a_4 = 2.7125

So $X_4 = (a_3 + a_4) / 2$ = (2.7 + 2.7125)/2 = 2.70625

Now f (X_4) = - 0.0049958497 = - ve

Therefore the root lies in (2.70625 ; 2.7125)

V)Again let a_4 = 2.70625 and a_5 = 2.7125

So $X_5 = (a_4 + a_5) / 2$ = (2.70625 + 2.7125)/2 = 2.709375

Now f (X_5) = 0.051243988 = + ve

Therefore the approximate real root of the given equation is x= 2.71 up to two digits.

ITERATION METHOD

Let the given equation be $f(x) = 0$ and the value of x to be determined.

By using the Iteration method you can find the roots of the equation.

To find the root of the equation first we have to write equation like below

$$\mathbf{x = \phi(x)}$$

Let $x=x_0$ be an initial approximation of the required root α then the first approximation x_1 is given by $x_1 = \phi(x_0)$.

Similarly for second, third and so on. approximation

$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2)$$

$$x_4 = \phi(x_3)$$

Continuing in this manner ,we get

$$x_n = \phi(x_{n-1}) \;;n=1,2,....$$

In this method is convergent conditionally and the condition is that $|\boldsymbol{\phi^{I}(x)}|$**<1** in the neighbourhood of the real root of $\mathbf{x = x_0}$.

Which is known as Iteration Method.

1. Find the real root of the equation $x^3 + x^2 = 1$ by iteration method.

Solution:

Given equation $f(x)= x^3 + x^2 - 1 = 0$;

Now $f(0.7) = (0.7)^3+(0.7)^2-1= -0.167 = -$ ve

and $f(0.8) =(0.7)^3+(0.7)^2-1 = 0.152 = +$ ve

Hence the root lies between(0.7 ;0.8)

Also $f(x) =x^3 + x^2 - 1 = 0$

$\Rightarrow x^2 (x + 1) = 1$

$\Rightarrow x^2 = \frac{1}{X+1}$

$\Rightarrow x = \frac{1}{\sqrt{(X+1)}} = \phi(x)$ Where $\phi(x)= \frac{1}{\sqrt{(X+1)}}$

Now $\phi^1(x)=-\frac{1}{2(X+1)\frac{3}{2}}$

Let the initial approximation be $x_0 = 0.8$

Now $|\phi^l(x_0)|=|-\frac{1}{2(0.8+1)\frac{3}{2}}|= 0.411035345 < 1$ in the nbd. of $x=x_0$.

Hence Iteration method is applicable.

Therefore Iteration method is $\mathbf{x_n = \phi(x_{n-1})}$;n=1,2,....

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{(X_0+1)}} = \frac{1}{\sqrt{(0.8+1)}} = 0.745355992$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{(X_1+1)}} = 0.756933958$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{(X_2+1)}} = 0.754435787$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{(X_3+1)}} = 0.754972723$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{(X_4+1)}} = 0.754857222$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{(X_5+1)}} = 0.754882063$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{(X_6+1)}} = 0.75487672$$

$$x_8 = \phi(x_7) = \frac{1}{\sqrt{(X_7+1)}} = 0.754877869$$

$$x_9 = \phi(x_8) = \frac{1}{\sqrt{(X_8+1)}} = 0.754877622$$

$$x_{10} = \phi(x_{10}) = \frac{1}{\sqrt{(X_9+1)}} = 0.754877675$$

$$x_{11} = \phi(x_{11}) = \frac{1}{\sqrt{(X_{10}+1)}} = 0.754877664$$

$$x_{12} = \phi(x_{12}) = \frac{1}{\sqrt{(X_{11}+1)}} = 0.754877666$$

Here $x_{12} \approx x_{11}$

Therefore the approximate real root of the given equation is $x=0.754877666$.

2. Find the real root of the equation 3x=cosx +1 by iteration method.

Solution:

Given equation $f(x)= 3x-\cos x-1 = 0$;

Now $f(0.6) = 3(0.6)-\text{Cos}(0.6)-1= -0.025335614$ = - ve

and $f(0.7) = 3(0.7)-\text{Cos}(0.7)-1= 0.335157812$ = + ve

Hence the root lies between(0.6;0.7)

Also $f(x) =3x-\cos x - 1 = 0$

=> $3x =\cos x+ 1$

=> $x = \frac{\text{Cos}\,x + 1}{3} = \phi(x)$ Where $\phi(x)=\frac{\text{Cos}\,x + 1}{3}$

Let the initial approximation be $x_0 = 0.6$

Now $|\phi'(x_0)|=|-\frac{\text{Sin}x_0}{3}|=|-\frac{\text{Sin}\,0.6}{3}| < 1$ in the nbd. of $x=x_0$.

Hence Iteration method is applicable.

Therefore Iteration method is $\mathbf{x_n = \phi(x_{n-1})}$;n=1,2,....

So $x_1 = \phi(x_0) = \frac{Cos\, x_0 + 1}{3} = \frac{Cos\, 0.6 + 1}{3}$ =0.608445205

$x_2 = \phi(x_1) = 0.606845906$

$x_3 = \phi(x_2) = 0.6071050271$

$x_4 = \phi(x_3) = 0.607092401$

$x_5 = \phi(x_4) = 0.607103406$

$x_6 = \phi(x_5) = 0.607101313$

$x_7 = \phi(x_6) = 0.607101711$

$x_8 = \phi(x_7) = 0.607101636$

$x_9 = \phi(x_8) = 0.60710165$

$x_{10} = \phi(x_{10}) = 0.607101647$

$x_{11} = \phi(x_{11}) = 0.607101648$

$x_{12} = \phi(x_{12}) = 0.607101648$

Here $x_{11} \approx x_{12}$

Therefore the approximate real root of the given equation is x=0.607101648.

3.Solve x=0.21 Sin(0.5+x) by iteration method starting with x=0.12.

Solution:

Given equation $x=0.21\,Sin(0.5+x)$

=> $x=0.21\,Sin(0.5+x) = \phi(x)$ Where $\phi(x)= 0.21Sin(0.5+x)$

Let the initial approximation be $x_0 = 0.12$

Now $|\phi^{l}(x_0)|=|0.21\,Cos(0.5+x_0)|$

$=|0.21\,Cos(0.5+0.12)|= 0.099994145< 1$

in the nbd. of $x=x_0$.

Hence Iteration method is applicable.

Therefore Iteration method is $\mathbf{x_n = \phi(x_{n-1})}$;n=1,2,....

So $x_1 = \phi(x_0) = 0.21 \sin(0.5+x_0) = 0.21 \sin(0.5+0.12)$

$= 0.002272374338$

$x_2 = \phi(x_1) = 0.21 \sin(0.5+x_1) = 0.001840900823$

$x_3 = \phi(x_2) = 0.21 \sin(0.5+x_2) = 0.001839319451$

$x_4 = \phi(x_3) = 0.21 \sin(0.5+x_3) = 0.001839313655$

$x_5 = \phi(x_4) = 0.21 \sin(0.5+x_4) = 0.001839313634$

$x_6 = \phi(x_5) = 0.21 \sin(0.5+x_5) = 0.001839313634$

Here $x_5 \approx x_6$

Therefore the approximate real root of the given equation is $x=0.001839313634$

4. Find the real root of the equation $e^x Tanx = 1$ by iteration Method .

Solution:

Given equation f(x)= $e^x \text{Tanx} - 1$ = 0;

Now f(0.5) = -0.099299464 = - ve

and f(0.7) = 0.24657854 = + ve

Hence the root lies between(0.5;0.6)

Also f(x) = $e^x \text{Tanx} - 1$ = 0

=> Tan x = $\frac{1}{e^x}$

=> x = $Tan^{-1}(e^{-x}) = \phi(x)$ Where $\phi(x)= Tan^{-1}(e^{-x})$

Let the initial approximation be $x_0 = 0.5$

Now $|\phi^{I}(x_0)| = |-\dfrac{e^{-x}}{1+e^{-2x}}| < 1$ in the nbd. of $x=x_0$.
Hence Iteration method is applicable.
Therefore Iteration method is $\mathbf{x_n = \phi(x_{n-1})}$;n=1,2,....

So $x_1 = \phi(x_0) = Tan^{-1}(e^{-x_0}) = Tan^{-1}(e^{-0.5}) = 0.54207623$

$x_2 = \phi(x_1) = Tan^{-1}(e^{-x_1}) = 0.525375363$

$x_3 = \phi(x_2) = Tan^{-1}(e^{-x_2}) = 0.534022624$

$x_4 = \phi(x_3) = Tan^{-1}(e^{-x_3}) = 0.530241889$

$x_5 = \phi(x_4) = Tan^{-1}(e^{-x_4}) = 0.531892929$

$x_6 = \phi(x_5) = Tan^{-1}(e^{-x_5}) = 0.531171549$

$x_7 = \phi(x_6) = Tan^{-1}(e^{-x_6}) = 0.531406667$

$x_8 = \phi(x_7) = Tan^{-1}(e^{-x_7}) = 0.531349002$

$x_9 = \phi(x_8) = Tan^{-1}(e^{-x_8}) = 0.531409141$

$x_{10} = \phi(x_{10}) = Tan^{-1}(e^{-x_9}) = 0.531382769$

$x_{11} = \phi(x_{11}) = Tan^{-1}(e^{-x_{10}}) = 0.531394346$

$x_{12} = \phi(x_{12}) = Tan^{-1}(e^{-x_{11}}) = 0.531389332$

Here $x_{11} \approx x_{12}$

Therefore the approximate real root of the given equation is x=0.53139.

5. Find the real root of the equation $x^3+x-1=0$ by iteration Method .

Sol:

Given equation $f(x) = x^3+x-1=0$

Now $f(0.6)= -0.184 = -$ Ve

And $f(0.7) = 0.043 = +$ Ve

Hence the root lies between (0.6 , 0.7)

Also $x^3+x-1=0 \Rightarrow x(x^2+1)=1$

$\Rightarrow x = \frac{1}{x^2+1} = \phi(x)$ Where $\phi(x)= \frac{1}{x^2+1}$

Let the initial approximation be $x_0 = 0.7$

Now $|\phi^{l}(x_0)| = \left|-\frac{2x}{(1+x^2)^2}\right| < 1$ in the nbd. of $x=x_0$.

Hence Iteration method is applicable.

Therefore Iteration method is $\mathbf{x_n = \phi(x_{n-1})}$;n=1,2,....

So $x_1 = \phi(x_0) = \dfrac{1}{x_0^2 + 1} = \dfrac{1}{0.7^2 + 1} = 0.671140939$

$x_2 = \phi(x_1) = 0.689450638$

$x_3 = \phi(x_2) = 0.677808858$

$x_4 = \phi(x_3) = 0.685201434$

$x_5 = \phi(x_4) = 0.680503106$

$x_6 = \phi(x_5) = 0.683487532$

$x_7 = \phi(x_6) = 0.681591146$

$x_8 = \phi(x_7) = 0.682795903$

$x_9 = \phi(x_8) = 0.682030427$

$x_{10} = \phi(x_{10}) = 0.682516751$

$x_{11} = \phi(x_{11}) = 0.682207761$

$x_{12} = \phi(x_{12}) = 0.682404073$

Here $x_{11} \approx x_{12}$

Therefore the approximate real root of the given equation is $x=0.6823$ up to four decimal places .


NEWTON-
RAPHSON
METHOD

Let x_0 be the an approximate value of a root of the equation $f(x)=0$ and let (x_0+h) be the exact value of the corresponding root where h is very small quantity. Then $f(x_0+h)=0$ expanding by using Taylor´s theorem we get

$$f(x_0)+hf^1(x_0)+(h^2/2)\, f^{11}(x_0)+\dots =0$$

Since h is very small ,neglecting Second and higher order terms and taking the first approximation we have

$$f(x_0)+hf^1(x_0) = 0$$

$$\Rightarrow hf^1(x_0) = - f(x_0)$$

$$\Rightarrow h = - f(x_0)/ f^1(x_0), \text{ provided } f^1(x_0) \neq 0$$

$$\Rightarrow x_1 = x_0+h = x_0- [f(x_0)/ f^1(x_0)]$$

$$\Rightarrow \mathbf{x_1 = x_0- [f(x_0)/ f^1(x_0)] \ldots\ldots\ldots\ldots\ldots\ (1)}$$

equation (1) gives the improved value of the root over the previous (1).

Now putting x_1 for x_0 and x_2 for x_1 in equation (1) we get

$$\mathbf{x_2 = x_1 - [\, f(x_1) / f^1(x_1) \,]}$$

Continuing in this manner ,we get

$$\mathbf{X_{n+1} = x_n - [\, f(x_n) / f^1(x_n) \,]},$$

n=0,1,2,3,....

This is known as **Newton-Raphson** method.

The process can be continued till the desired accuracy has been achieved.

1. Find the real root of the equation $x^3-2x-5=0$ by Newton-Raphson method.

Solution:

Given equation $f(x) = \mathbf{x^3-2x-5=0}$

Now $f(2) = 2^3-2(2)-5 = -1 = -$ ve

and $f(2.1) = (2.1)^3-2(2.1)-5 = 0.061 = +$ ve

Hence the root lies between $(2, 2.1)$

Let the initial approximation be $x_0 = 2.1$

Also $f(x) = \mathbf{x^3-2x-5=0}$

Now $f^1(x) = 3x^2-2$

and $f^1(x_0) = 3(2.1)^2-2=11.23 \neq 0.$

By **Newton-Raphson** method we have

$$\mathbf{X_{n+1} = x_n - [\ f(x_n)/\ f^1(x_n)\ ]\ ,\ n=0,1,2,3,....}$$

So $x_1 = x_0 - [\ f(x_0)/\ f^1(x_0)\ ]$

$x_1 = 2.1 - [\ ((2.1)^3 - 2(2.1) - 5)/(3(0.1)^2 - 2)\ ]$

$= 2.094568121$

$x_2 = x_1 - [\ f(x_1)/\ f^1(x_1)\ ] = 2.094551482$

$x_3 = x_2 - [\ f(x_2)/\ f^1(x_2)\ ] = 2.094551482$

Here $x_2 \approx x_3$

Hence the approximate real root of the given equation is x= 2.094551482.

2.Find the real root of the equation 3x=cosx +1 by Newton-Raphson method.

Solution:

Given equation $f(x)= 3x-\cos x-1 = 0$;

Now $f(0.6) = 3(0.6)-\cos(0.6)-1= -0.025335614$ = - ve

and $f(0.7) = 3(0.7)-\cos(0.7)-1= 0.335157812$ = + ve

Hence the root lies between(0.6;0.7)

Let the initial approximation be $x_0 = 0.6$

Also $f(x)= 3x-\cos x-1$

Now $f^1(x) = 3+ \sin x$

and $f^1(x_0) = 3+\sin 0.6 =3.564642473 \neq 0$.

By **Newton-Raphson** method we have

$$X_{n+1} = x_n- [f(x_n)/ f^1(x_n)] , n=1,2,3,....$$

So $x_1 = x_0 - [\ f(x_0)/\ f^1(x_0)\ ]$

$x_1 = 0.6 - [(3x0.6-\mathrm{Cos}(0.6)-1)/(3+\mathrm{Sin}(0.6))]$

$= 0.607107477$

$x_2 = x_1 - [\ f(x_1)/\ f^1(x_1)\ ] = 0.607101648$

$x_3 = x_2 - [\ f(x_2)/\ f^1(x_2)\ ] = 0.607101648$

Here $x_2 \approx x_3$

Hence the approximate real root of the given equation is $x = 0.607101648$.

3.Find the real root of the equation $x \log_{10}x=1.2$ by Newton-Raphson method.

Solution:

Given equation $f(x)= \mathbf{x \log_{10}x-1.2} = 0$;

Now $f(2.7) = 2.7 \log_{10}2.7-1.2 = -0.035317836 = -$ ve

and $f(2.8) = 2.8\log_{10}2.8-1.2 = 0.052042487 = +$ ve

Hence the root lies between(2.7,2.8)

Let the initial approximation be $x_0 = 2.7$

Also $f(x)= x \log_{10}x-1.2$

Now $f^1(x) = \log_{10}x + x(1/x)\log_N e = \log_{10}x + 0.43429$

and $f^1(x_0) = \log_{10}2.7+0.43429 =0.865653764 \neq 0.$

By **Newton-Raphson** method we have

$$\mathbf{X_{n+1} = x_n- [f(x_n)/ f^1(x_n)] , n=1,2,3,....}$$

So $x_1 = x_0 - [\ f(x_0) / f^1(x_0)\ ]$

$x_1 = 2.7 - [(2.7 \log_{10} 2.7 - 1.2\ ) / (\ \log_{10} 2.7 + 0.43429\ )]$

$= 2.740799033$

$x_2 = x_1 - [\ f(x_1) / f^1(x_1)\ ] = 2.740646097$

$x_3 = x_2 - [\ f(x_2) / f^1(x_2)\ ] = 2.740646096$

Here $x_2 \approx x_3$

Hence the approximate real root of the given equation is x= 2.740646096.

4.Find the real root of the equation x^2-5x+2=0 by Newton-Raphson's method.

Sol: Given equation $f(x) = x^2-5x+2=0$

Now $f(0.4) = (0.4)^2-5(0.4)+2 = 0.16$ = +ve

And $f(0.5) = (0.5)^2-5(0.5)+2 = -0.25$ = -ve

Therefore the real root of the given equation lies between (0.4 , 0.5)

Let us take the origin $x_0 = 0.4$

By Newton-Raphson method we have

$$X_{n+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)} \right], \quad n=1,2,3,....$$

So $x_1 = x_0 - [\frac{f(x_0)}{f^1(x_0)}] = x_0 - [\frac{{x_0}^3 - 3x_0 - 5}{3x_0^2 - 3}]$

$= 2.3 - [\frac{2.3^3 - 3(2.3) - 5}{3(2.3)^2 - 5}]$

$= 2.275436983$

$x_2 = x_1 - [\frac{f(x_1)}{f^1(x_1)}] = 2.279707226$

$x_3 = x_2 - [\frac{f(x_2)}{f^1(x_2)}] = 2.27888909$

$x_4 = x_3 - [\frac{f(x_3)}{f^1(x_3)}] = 2.279043314$

Here $x_3 \approx x_4$

Hence the approximate real root of the given equation is $x = 2.279$.

So $x_1 = x_0 - [\frac{f(x_0)}{f^1(x_0)}] = x_0 - [\frac{{x_0}^2 - 5x_0 + 2}{2x_0 - 5}]$

$$= 0.4 - [\frac{0.4^2 - 5(0.4) + 2}{2(0.4) - 5}]$$

$$= 0.438095238$$

$x_2 = x_1 - [\frac{f(x_1)}{f^1(x_1)}] = 0.438447157$

$x_3 = x_2 - [\frac{f(x_2)}{f^1(x_2)}] = 0.438447187$

$x_4 = x_3 - [\frac{f(x_3)}{f^1(x_3)}] = 0.438447187$

Here $x_3 \approx x_4$

Hence the approximate real root of the given equation is $x = 0.438447187$.

5. Find a real root of the equation x^3-3x-5=0 by Newton-Raphson method.

Sol :

Given equation $f(x)= x^3-3x-5=0$

Now $f(2.2) = (2.2)^3-3(2.2)-5 = -0.952$ = -ve

And $f(2.3) = (2.3)^3-3(2.3)-5 = 0.267$ = +ve

Therefore the real root of the given equation lies between (2.2 ,2.3)

Let us take the origin $x_0 = 2.3$

By Newton-Raphson method we have

$$X_{n+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)}\right],\ n=1,2,3,....$$

So $x_1 = x_0 - \left[\frac{f(x_0)}{f^1(x_0)}\right] = x_0 - \left[\frac{{x_0}^3 - 3x_0 - 5}{3x_0^2 - 3}\right] = 2.3 - \left[\frac{2.3^3 - 3(2.3) - 5}{3(2.3)^2 - 5}\right]$

$=2.275436983$

$x_2 = x_1 - \left[\frac{f(x_1)}{f^1(x_1)}\right] = 2.279707226$

$x_3 = x_2 - \left[\frac{f(x_2)}{f^1(x_2)}\right] = 2.27888909$

$x_4 = x_3 - \left[\frac{f(x_3)}{f^1(x_3)}\right] = 2.279043314$

Here $x_3 \approx x_4$

Hence the approximate real root of the given equation is $x = 2.279$.

6.**Find a real root of the equation x^4-x-10=0 correct to three decimal places by Newton-Raphson method.**

Sol : Given equation $f(x)= x^4-x-10=0$

Now $f(1.8) = (1.8)^4-(1.8)-10 = -1.3024$ = -ve

And $f(1.9) = (1.9)^4-(1.9)-10 = 1.1321$ = +ve

Therefore the real root of the given equation lies between

(1.8 ,1.9)

Let us take the origin $x_0 = 1.8$

By Newton-Raphson method we have $\mathbf{X_{n+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)}\right]}$ **, n=1,2,3,....**

So $x_1 = x_0 - \left[\frac{f(x_0)}{f^1(x_0)}\right] = x_0 - \left[\frac{{x_0}^4 - x_0 - 10}{4x_0^3 - 1}\right] = 1.8 - \left[\frac{1.8^4 - (1.8) - 10}{4(1.8)^3 - 1}\right]$

$= 1.858330348$

$x_2 = x_1 - [\frac{f(x_1)}{f^1(x_1)}]$

$= 1.855590855$

$x_3 = x_2 - [\frac{f(x_2)}{f^1(x_2)}]$

$= 1.855584529$

$x_4 = x_3 - [\frac{f(x_3)}{f^1(x_3)}]$

$= 1.855584529$s

Here $x_3 \approx x_4$

Hence the approximate real root of the given equation is

$x = 1.855584529.$

7. Using Newton-Raphson method , establish the iterative formula $x_{n+1} = (1/3)[2\, x_n + (N/x_n^2)]$ to calculate the cube root of N. Hence deduce the value of cube root of 12.

Solution:

Let $x = \sqrt[3]{N} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

Let $f(x) = x^3 - N$. Then $f^1(x) = 3x^2$.

By **Newton-Raphson** method we have

$$X_{n+1} = x_n - [\, f(x_n) / f^1(x_n)\,]\,,\ n=1,2,3,\ldots.$$

$$= x_n - [\, (x_n^3 - N)/3x_n^2)\,]$$

$$= (2x_n^3 + N) / 3x_n^2$$

$$X_{n+1} = (1/3)\, [\, 2x_n + (N/x_n^2)\,] \text{ for } n=0,1,2,\ldots.$$

Let $x = \sqrt[3]{12} \Rightarrow x^3 = 12 \Rightarrow x^3 - 12 = 0$

Let $f(x) = x^3 - 12$. Then $f^1(x) = 3x^2$.

By **Newton-Iteration** method we have

$$\mathbf{X_{n+1}} = (1/3)\,[\, 2x_n + (N/x_n^{\,2})\,] \text{ for } n=0,1,2,....$$

Let the initial approximation be $x_0 = 2.3$ and $N=12$

$X_1 = (1/3)[\, 2x_0 + (12/x_0^{\,2})\,] = (1/3)[2(2.3)+(12/2.3^2)]$

$= 2.289477001$

$X_2 = (1/3)[\, 2x_1 + (12/x_1^{\,2})\,] = 2.289428486$

$X_3 = (1/3)[\, 2x_2 + (12/x_2^{\,2})\,] = 2.289428485$

$X_4 = (1/3)[\, 2x_3 + (12/x_3^{\,2})\,] = 2.289428485$

Here $X_3 \approx X_4$

Hence the approximate value of $\sqrt[3]{12} = 2.289428485$.

8. Using Newton-Raphson method , establish the iterative formula $x_{n+1} = (1/2)[\, x_n + (N/x_n)\,]$ to calculate the Square root of N. Hence deduce the value of Square root of 2.

Solution:

Let $x = \sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0$

Let $f(x) = x^2 - N$. Then $f^1(x) = 2x$.

By **Newton-Raphson** method we have

$$\mathbf{X_{n+1} = x_n - [\, f(x_n) / f^1(x_n)\,]}$$

$$= x_n - [\, (x_n^2 - N)/2x_n)\,]$$

$$= (x_n^2 + N)\,/\,2x_n$$

$$\mathbf{X_{n+1}} = (1/2)\,[\, x_n + (N/x_n)\,] \text{ for } n=0,1,2,\ldots.$$

Let $x = \sqrt{2} \Rightarrow x^2 = 2 \Rightarrow x^2 - 2 = 0$

Let $f(x) = x^2 - 2$. Then $f^1(x) = 2x$.

By **Newton-Iteration** method we have

$$\mathbf{X_{n+1}} = (1/2)\,[\, x_n + (N/x_n)\,] \text{ for } n=0,1,2,....$$

Let the initial approximation be $x_0 = 1.4$ and $N=2$

$X_1 = (1/2)[\, x_0 + (2/x_0)\,] = (1/2)[1.4+(2/1.4)]$

$= 1.414285714$

$X_2 = (1/2)[\, x_1 + (2/x_1)\,] = 1.414213564$

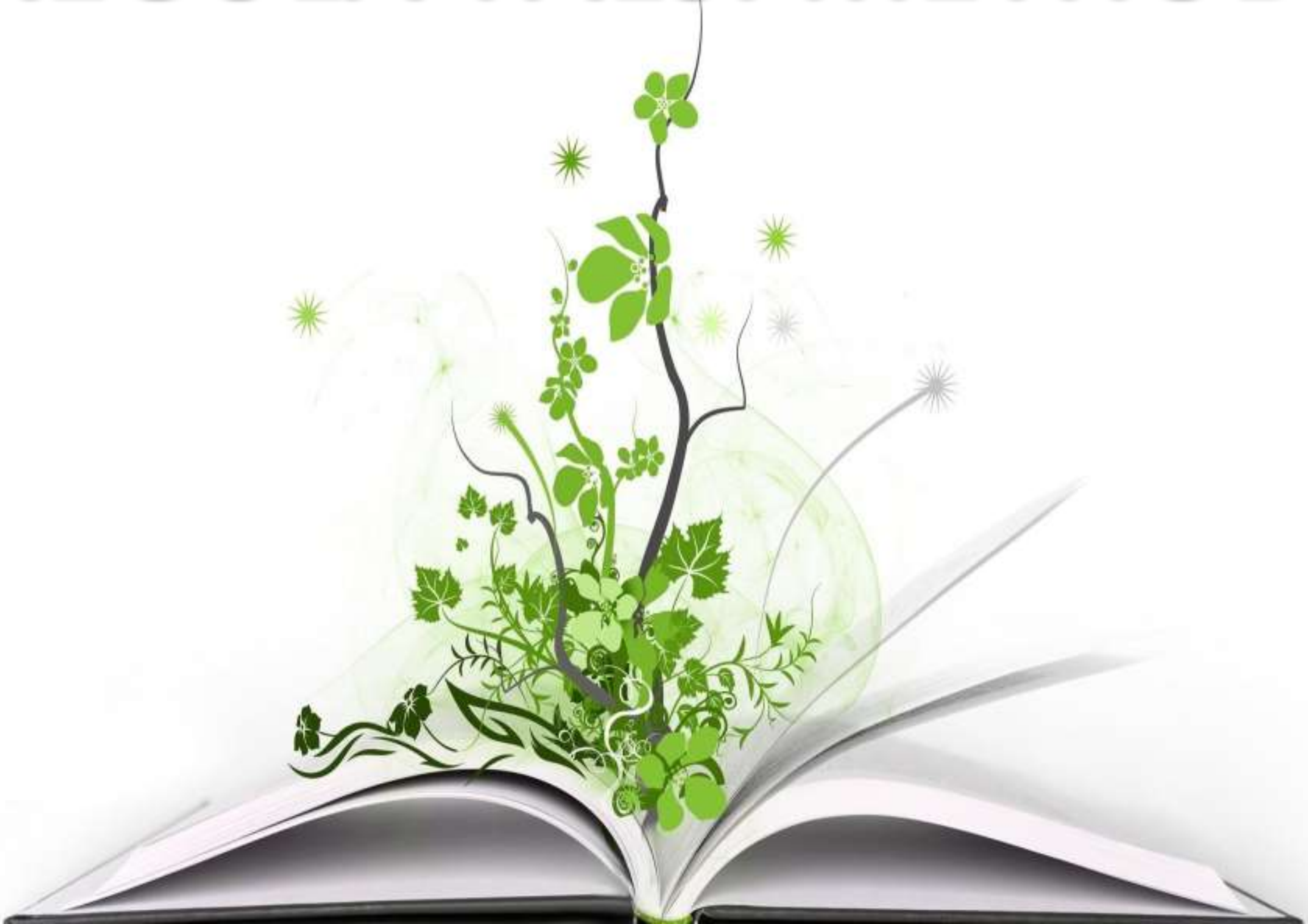
$X_3 = (1/2)[\, x_2 + (2/x_2)\,] = 1.414213562$

$X_4 = (1/2)[\, x_3 + (2/x_3)\,] = 1.414213562$

Here $X_3 \approx X_4$

Hence the approximate value of $\sqrt{2} = 1.414213562$.

REGULA-FALSI METHOD



Consider the equation $f(x)=0$ and let a,b be two values of x such that f(a) and f(b) are of opposite sign. Also let $a<b$. The graph of $y=f(x)$ will meet the x-axis at the some point between a and b .The equation of the line joining the two points (a , f(a)), (b , f(b)) is

$y-f(a) = [(f(b)-f(a))/(b-a)] / (x-a)$ .... (1)

In the small interval (a , b),the graph of the function can be considered as a straight line. So the x-co-ordinate of the point of intersection of the chord joining (a,f(a)),(b,f(b)) with the axis will give an approximate value of the root.

So by putting y=0 in equation (1) we get

-f(a) = [(f(b)-f(a))/(b-a)] / (x-a)

$\Rightarrow$x= (af(b)-bf(a)) / (f(b)-f(a))

If we take (a,b)as ($a_{1,}$ b_1) then the first approximation root is

x_1= ($a_1f(b_1)-b_1f(a_1)$) / ($f(b_1)-f(a_1)$) .... (2)

If $f(a_1)$and $f(x_1)$ are of opposite signs then the root lies between a_1 and x_1 ; otherwise it lies between x_1 and b_1 .We rename the interval in which the root lies as (a_2 ,b_2).

Then the next approximation root is

$x_2 = (a_2 f(b_2)-b_2 f(a_2)) / (f(b_2)-f(a_2))$ .... (3)

Proceeding in this way we get

$x_n = (a_n f(b_n)-b_n f(a_n)) / (f(b_n)-f(a_n))$

This method applied repeatedly till the desired accuracy is obtained.

This method is known as falsi position method or Regula-Falsi method or method of chord.

1. Find areal root of the equation $f(x)=x^3-2x-5=0$ by the method of false position up to three places of decimals.

Solution:

Given equation $f(x)=x^3-2x-5=0$

Now $f(2)=(2)^3-2(2)-5= -1 = -ve$

And $f(2.1)=(2.1)^3-2(2.1)-5= 0.061 = +ve$

Therefore the root lies between (2,2.1)

By Regula-Falsi method, we

$$X_n = [\, a_n f(b_n)-b_n f(a_n) \,] / [f(b_n)-f(a_n) \,]$$

I) Let $a_0 = 2$ & $f(a_0) = -1$

$b_0 = 2.1$ & $f(b_0) = 0.061$

Now $x_0 = [a_0 f(b_0)-b_0 f(a_0)] / [f(b_0)-f(a_0)]$

$= [\, 2x0.061+1x2.1] / [0.061+1]$

$=2.094250707$

So $f(x_0)=(2.094250707)^3-2(2.094250707)-5$

$= -0.00335650926 = -ve$

Therefore the root lies between (2.094250707 , 2.1)

II) Let $a_1=2.094250707$ & $f(a_1) = -0.00335650926$

$b_1 = 2.1$ & $f(b_1) = 0.061$

Now $x_1 = [a_1f(b_1) - b_1f(a_1)] / [f(b_1)-f(a_1)]$

$=[2.094250707 \text{x} 0.061+2.1 \text{x} 0.00335650926]/[0.061+0.00335650926]$

$= 2.094550561$

So $f(x_1) = (2.094550561)^3-2(2.094550561)-5$

$= -0.00001027482 = -ve$

Therefore the root lies between (2.094550561 , 2.1)

III) Let $a_2=2.094550561$ & $f(a_2) = - 0.00001027482$

$b_2 = 2.1$ & $f(b_2) = 0.061$

Now $x_2 = [a_2f(b_2)-b_2f(a_2)] / [f(b_2)-f(a_2)]$

$= 2.094551479$

So $f(x_2)=(2.094551479)^3-2(2.094551479)-5$

$= - 0.00000003121= -$ ve

Which is nearer to Zero.

Therefore the approximate real root of the given equation is $x = 2.094551479$.

RAMANUJAN'S METHOD

Srinivasa Ramanujan described an iterative method which can be used to determine the smallest root of the equation $f(x)=0$ ...... (1)

Where $f(x)=1-(a_1x+a_2x^2+a_3x^3+a_4x^4+......)$ ...... (2)

For smaller values of x , we can write

$[\ 1-(a_1x+a_2x^2+a_3x^3+a_4x^4+......)\ ]^{-1} = b_1+b_2x+b_3x^2+.......$ ....(3)

Expanding the LHS of equation (3) by using Binomial theorem, we get

$1+(a_1x+a_2x^2+a_3x^3+a_4x^4+......)+(a_1x+a_2x^2+a_3x^3+a_4x^4+......)^2+....= b_1+b_2x+b_3x^2+.......$ ......(4)

Comparing the coefficients of like powers of x on bothsides of (4),we get

$b_1 = 1$

$b_2 = a_1 = a_1 b_1$

$b_3 = a_1^2 + a_2 = a_1 b_2 + a_2 b_1$

……………………………………………………

$b_n = a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_{n-1} b_1$, $n = 2, 3, \dots$

Ramanujan states that the successive convergent, i.e.,

b_n / b_{n+1}

approaches a root of the equation (2)

1. Find the smallest root of the equation $f(x)=x^3-6x^2+11x-6 = 0$ using Ramanujan's method.

Solution :

Given $f(x)=x^3-6x^2+11x-6 = 0$ .... (1)

We write the given equation (1) as follows:

$-6[\ 1-(\ (11x-6x^2+x^3)/6)\ ] = 0$

$\Rightarrow [\ 1-(\ (11x-6x^2+x^3)/6)\ ]^{-1} = b_1 + b_2x + b_3x^2 + \ldots\ldots$ ....(2)

$\Rightarrow [\ 1-(a_1x + a_2x^2 + a_3x^3 + \ldots\ldots)\ ]^{-1} = b_1 + b_2x + b_3x^2 + \ldots\ldots$

Where $a_1 = (11/6)$; $a_2 = -1$; $a_3 = (1/6)$; $a_4=a_5=\ldots\ldots =0$

Comparing the like coefficients, we have

$b_1 = 1$

$b_2 = a_1 = 11/6$

$b_3 = a_1b_2+a_2b_1 = (11/6)(11/6) + (-1)(1) = 85/36$

$b_4 = a_1b_3+a_2b_2+a_3b_1 = (11/6)(85/36)+(-1)(11/6)+(1/6)(1)$
$=575/216$

$b_5 = a_1b_4+a_2b_3+a_3b_2+a_4b_1 = 3661/1296$

$b_6 = a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1 = 22631/1296$

Therefore

$b_1/b_2 = 1/6 = 0.1666666667$

$b_2/b_3 = (11/6) / (85/36) = 0.7764705882$

$b_3/b_4 = (85/36) / (575/216) = 0.8869565217$

$b_4/b_5 = (575/216) / (3661/1296) = 0.9423654739$

$b_5/b_6 = (3661/216) / (22631/1296) = 0.9706155274.$

We observe that, the successive convergents approach the root.

Hence the smallest root of the given equation is $x=1$.

2.Find the smallest root of the equation $x\ e^x = 1$ using Ramanujan's method.

Solution :

Given $\mathbf{x\ e^x = 1}$ .... (1)

Expanding e^x in ascending powers of x and simplifying we can rewrite as

$X(1 + x + (x^2/2) + (x^3/6) + (x^4/24) + \dots\dots\dots\dots = 1$

$\Rightarrow x + x^2 + (x^3/2) + (x^4/6) + (x^5/24) + \dots\dots\dots\dots = 1$

Hence we write

$[\ 1 - (x + x^2 + (x^3/2) + (x^4/6) + (x^5/24) + \dots\dots\dots\dots\ )\ ]^{-1} = b_1 + b_2x + b_3x^2 + \dots\dots\dots$ ....(2)

$\Rightarrow [\ 1 - (a_1x + a_2x^2 + a_3x^3 + \dots\dots\ )\ ]^{-1} = b_1 + b_2x + b_3x^2 + \dots\dots\dots$

Where $a_1 = 1$; $a_2 = 1$; $a_3 = (1/2)$; $a_4 = (1/6)$: $a_5 = (1/24)$:.....

Comparing the like coefficients, we have

$b_1 = 1$

$b_2 = a_1 = 1$

$b_3 = a_1b_2+a_2b_1 = 1+1 = 2$

$b_4 = a_1b_3+a_2b_2+a_3b_1 = 2+1+(1/2) = 7/2$

$b_5 = a_1b_4+a_2b_3+a_3b_2+a_4b_1 = (7/2)+2+(1/2)+(1/6) = 37/6$

$b_6 = a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1 = (37/6)+(7/2)+1+(1/6) +(1/24)$
$=261/24$

Therefore

$b_1/b_2 = 1$

$b_2/b_3 = 1/2= 0.5$

$b_3/b_4 = 2x(2/7)= 4/7= 0.571428571$

$b_4/b_5 = (7/2)x(6/37) = 0.567567567$

$b_5/b_6 = (37/6)x(24/261) = 0.567049808.$

Hence the approximate root of the given equation is x=0.567 up to three decimal places.

3.Find the smallest root of the equation $1- x + (x^2/2!^2) – (x^3/3!^2) + (x^4/4!^2) - .... = 0$ using Ramanujan’s method.

Solution :

Given **$1- x + (x^2/2!^2) – (x^3/3!^2) + (x^4/4!^2) - .... = 0$**

Let $[\ \mathbf{1- (x - (x^2/2!^2) + (x^3/3!^2) - (x^4/4!^2) + ...\ )}\ ]^{-1} = b_1 + b_2x + b_3x^2 + ...$

$\Rightarrow\ [\ 1- (a_1x + a_2x^2 +a_3x^3 + ...\ )\ ]^{-1} = b_1 + b_2x + b_3x^2 + ...$

Where $a_1 = 1$; $a_2 = 1/(2!)^2$; $a_3 = 1/(3!)^2$; $a_4= 1/(4!)^2$:

$a_5= 1/(5!)^2$:.....

Comparing the like coefficients, we have

$b_1 = 1$

$b_2 = a_1 = 1$

$b_3 = a_1b_2+a_2b_1 = 1- 1/(2!)^2 = 3/4$

$b_4 = a_1b_3+a_2b_2+a_3b_1 = (3/4)- (1/(2!)^2)+((1/3!)^2) = 19/36$

$b_5 = a_1b_4+a_2b_3+a_3b_2+a_4b_1 = 211/576$

Therefore

$b_1/b_2 = 1$

$b_2/b_3 = 4/3 = 1.333333333$

$b_3/b_4 = 27/19 = 1.421052632$

$b_4/b_5 = (19/36)/(211/576) = 1.440758294$

Hence the approximate root of the given equation is $x=1.440758294$.

Thank you!